Mathematics

# Planning Guide

## Grade 5 *Volume*

## Shape and Space (Measurement)
Specific Outcome 4

**This Planning Guide can be accessed online at:**
http://www.learnalberta.ca/content/mepg5/html/pg5_volume/index.html

# Table of Contents

# **Planning Guide:** *Grade 5 Volume*

**Strand:** Shape and Space (Measurement)
**Specific Outcome:** 4

This *Planning Guide* addresses the following outcomes from the Program of Studies:

| **Strand:** Shape and Space (Measurement) | |
|---|---|
| **Specific Outcome:** | 4. Demonstrate an understanding of volume by:<br>• selecting and justifying referents for $cm^3$ or $m^3$ units<br>• estimating volume, using referents for $cm^3$ or $m^3$<br>• measuring and recording volume ($cm^3$ or $m^3$)<br>• constructing right rectangular prisms for a given volume. |

## Curriculum Focus

The changes to the curriculum targeted by this sample include:

- The general outcome focuses on using direct or indirect measurement to solve problems; whereas the previous math curriculum focused on using measurement concepts, appropriate tools and results of measurements to solve problems in everyday contexts.
- The specific outcome focuses on most of the same concepts related to volume as the previous math curriculum, including estimating, measuring, recording and constructing rectangular prisms for a given volume. The previous math curriculum also included ordering containers by volume, which is not included in the present curriculum.
- The specific outcome includes referents for the units $cm^3$ or $m^3$; whereas the previous math curriculum does not.

## What Is a Planning Guide

**Planning Guides** are a tool for teachers to use in designing instruction and assessment that focuses on developing and deepening students' understanding of mathematical concepts. This tool is based on the process outlined in *Understanding by Design* by Grant Wiggins and Jay McTighe.

## Planning Steps

The following steps will help you through the Planning Guide:

## Step 1: Identify Outcomes to Address

### Guiding Questions

- What do I want my students to learn?
- What can my students currently understand and do?
- What do I want my students to understand and be able to do based on the Big Ideas and specific outcomes in the program of studies?

### Big Ideas

Van de Walle and Lovin (2006) define volume as "the amount of space that an object takes up" (p. 265). Volume is measured in units such as cubic centimetres ($cm^3$) or cubic metres ($m^3$), which are based on linear measures.

By estimating a measure first and then using measuring instruments to measure, students develop measurement sense. Estimation in measurement is defined as follows:

> "Measurement estimation is the process of using mental and visual information to measure or make comparisons without the use of measuring instruments. It is a practical skill" (Van de Walle and Lovin 2006, p. 278).

In using any type of measurement, such as length, area or volume, it is important to discuss the similarities between them in developing an understanding of the different measures. First identify the attribute to be measured, then choose an appropriate unit and finally compare that unit to the object being measured (NCTM 2000, p. 171). An attribute of an object is an aspect of that object that can be measured. "The measure of an attribute is a count of how many units are needed to fill, cover or match the attribute of the object being measured" (Van de Walle and Lovin 2006, p. 253). As with other attributes, it is important to understand the attribute of volume before measuring.

Key ideas in understanding the attribute of volume include:

- conservation—an object retains its size when the orientation is changed or it is rearranged by subdividing it in any way
- iteration—the repetitive use of an identical non-standard or standard units of volume to entirely fill or construct an object
- additivity—add the measures of the volume for each part of an object to obtain the measure of the entire object
- proportionality—there is an inverse relationship between the size of the unit used to measure volume and the number of units needed to measure the volume of a given object; i.e., the smaller the unit, the more you need to measure the volume of a given object
- transitivity—when direct comparison of two volumes is not possible, use a third item that allows comparison; e.g., to compare the volume of two boxes, find the volume of one box using non-standard or standard units and compare that measure with the volume of the other box (if A = B and B = C, then A = C)
- standardization—using standard units for measuring volume such as $cm^3$ and $m^3$ facilitates communication of measures globally
- unit/unit-attribute relations—units used for measuring volume must relate to volume; e.g., $cm^3$ must be used to measure volume and not cm or $cm^2$.

## Sequence of Outcomes from Program of Studies

See http://education.alberta.ca/teachers/core/math/programs.aspx for the complete program of studies.

| Grade 4 | Grade 5 | Grade 6 |
|---|---|---|
| **Specific Outcomes** | **Specific Outcomes** | **Specific Outcomes** |
| 3. Demonstrate an understanding of area of regular and irregular 2-D shapes by:<br>• recognizing that area is measured in square units<br>• selecting and justifying referents for the units $cm^2$ or $m^2$<br>• estimating area, using referents for $cm^2$ or $m^2$<br>• determining and recording area ($cm^2$ or $m^2$)<br>• constructing different rectangles for a given area ($cm^2$ or $m^2$) in order to demonstrate that many different rectangles may have the same area.<br><br>4. Describe and construct right rectangular and right triangular prisms. | 4. Demonstrate an understanding of volume by:<br>• selecting and justifying referents for $cm^3$ or $m^3$ units<br>• estimating volume, using referents for $cm^3$ or $m^3$<br>• measuring and recording volume ($cm^3$ or $m^3$)<br>• constructing right rectangular prisms for a given volume. | 3. Develop and apply a formula for determining the:<br>• perimeter of polygons<br>• area of rectangles<br>• volume of right rectangular prisms. |

## Step 2: Determine Evidence of Student Learning

### Guiding Questions

- What evidence will I look for to know that learning has occurred?
- What should students demonstrate to show their understanding of the mathematical concepts, skills and Big Ideas?

### Using Achievement Indicators

As you begin planning lessons and learning activities, keep in mind ongoing ways to monitor and assess student learning. One starting point for this planning is to consider the achievement indicators listed in the *Mathematics Kindergarten to Grade 9 Program of Studies with Achievement Indicators*. You may also generate your own indicators and use them to guide your observation of the students.

The following indicators may be used to determine whether or not students have met this specific outcome. Can students:

- rearrange a given object and determine that the volume of the object remains unchanged?
- use direct comparison to compare the volume of two objects?
- use nonstandard units to measure the volume of objects?
- make indirect comparisons by using nonstandard units to measure the volume of objects?
- measure the volume of an object using larger then smaller nonstandard units of measure to establish that the smaller the unit of measure, the more you need to measure the volume of a given object?
- identify the cube as the most efficient unit for measuring volume and explain why?
- provide a referent for a cubic centimetre and explain the choice?
- provide a referent for a cubic metre and explain the choice?
- determine which standard cubic unit is represented by a given referent?
- estimate the volume of a given 3-D object using personal referents?
- determine the volume of a given 3-D object, using manipulatives, and explain the strategy?
- construct a right rectangular prism for a given volume?
- construct more than one right rectangular prism for the same given volume?
- solve problems involving the volume of 3-D objects?

Sample behaviours to look for related to these indicators are suggested for some of the activities listed in **Step 3, Section C: Choosing Learning Activities** (p. 13).

## Step 3: Plan for Instruction

### Guiding Questions

- What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should I use?
- How will I meet the diverse learning needs of my students?

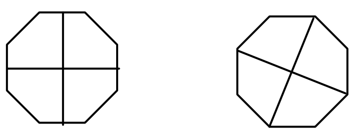
### A. Assessing Prior Knowledge and Skills

Before introducing new material, consider ways to assess and build on students' knowledge and skills related to counting. For example:

Have the students complete Part A (Estimating Area) and hand it in. Then provide them with Part B (Finding Area), along with centimetre rulers, centimetre grid paper, triangular or isometric dot paper, pattern blocks and other manipulatives they may wish to use.

**Part A: (Estimating Area)**

1. For the following pair of congruent shapes, decide if one of the four parts of the first figure has the same area as one of the four parts of the second figure. Explain your thinking.



2. Estimate the area of the following rectangle. Explain the referent you used in making the estimate.

**Part B: (Finding Area)**

1. Compare the areas of the following two rectangles. One rectangle is 4 cm by 12 cm. The other rectangle is 6 cm by 8 cm.

   a. Draw a diagram on cm grid paper and use words to explain your thinking.
   b. Explain how you could compare the areas of these two rectangles by rearranging one rectangle.

2. Explain why area is measured in square units.

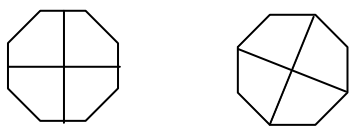
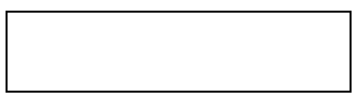
3. Lisa and her dad are building a rabbit hutch for their rabbits. They decide that the hutch will have an area of 18 $m^2$.

   a. Draw all the possible rectangular pens that have the area of 18 $m^2$ on cm grid paper. The sides of the rectangles must be measured in whole numbers. Each centimetre on the grid paper will represent 1 metre.
   b. Explain how you know that you have drawn all the possible rectangular pens.

4. Marie measured a pattern block design with red trapezoids and said that the measure of the area was 6 trapezoids. Jaden measured the same design and said the area was 18 green triangles. Could both be correct? Why or why not? Draw a diagram and use words to explain your thinking. Use the triangular dot paper provided.

   If a student appears to have difficulty with these tasks, consider further individual assessment, such as a structured interview, to determine the student's level of skill and understanding. See **Sample Structured Interview: Assessing Prior Knowledge and Skills** (p. 10).

## Sample Structured Interview: Assessing Prior Knowledge and Skills

| Directions | Date: | |
|---|---|---|
| | **Not Quite There** | **Ready to Apply** |
| Place the following figures before the student:<br><br>Say, "**For this pair of congruent shapes, decide if one of the four parts of the first figure has the same area as one of the four parts of the second figure. Explain your thinking.**" | Explains that the two parts do not have the same area because they do not have the same shape.<br>Or, says that the two parts do have the same area but is unable to explain why. | Explains that the two parts have the same area because they are each one-quarter of congruent shapes. |
| Place the following rectangle before the student:<br><br>Say, **"Estimate the area of this rectangle. Explain the referent you used in making the estimate. "** | Estimates the area but the estimate is inaccurate; i.e., different from the actual area by more than 20%.<br><br>Does not use a referent to make the estimate. | Uses a referent to estimate the area of the rectangle to within 20% of the actual area.<br><br>Explains how the referent is used to find the estimate. |
| Provide the student with centimetre grid paper.<br>Present the following problem to the student:<br>**"Compare the areas the two rectangles. One rectangle is 4 cm by 12 cm. The other rectangle is 6 cm by 8 cm.**<br>**a. Draw a diagram on cm grid paper and use words to explain your thinking.**<br>**b. Explain how you could compare the areas of these two rectangles by rearranging one rectangle."** | Draws the rectangles inaccurately on the grid paper.<br><br>Compares the areas of these two rectangles incorrectly with an unsuccessful attempt at rearranging one rectangle.<br>Or, compares the areas of these two rectangles correctly but is unable to rearrange one rectangle to show it has the same area as the other rectangle. | Draws the two rectangles accurately on the grid paper.<br><br>Compares the areas of the two rectangles correctly by rearranging one rectangle so that it can be superimposed perfectly onto the other rectangle, proving that the areas of the two rectangles are the same. |

<table>
<tr>
<td>Say, "Explain why area is measured in square units."</td>
<td>Provides a vague or no explanation of why area is measured in square units.</td>
<td>Provides a clear explanation of why area is measured in square units using appropriate mathematical language.<br>Example:<br>Squares do not leave gaps when used to cover a 2-D shape and there is only one shape for a square so everyone knows the shape of a square unit.</td>
</tr>
<tr>
<td>Provide the student with centimetre grid paper.<br>Present the following problem to the student:<br>"Lisa and her dad are building a rabbit hutch for their rabbits. They decide that the hutch will have an area of 18 $m^2$.<br>a. Draw all the possible rectangular pens that have the area of 18 $m^2$ on cm grid paper. The sides of the rectangles must be measured in whole numbers. Each centimetre on the grid paper will represent 1 metre.<br>b. Explain how you know that you have drawn all the possible rectangular pens."</td>
<td>Draws one or more but not all the possible rectangular pens.<br><br>Does not explain convincingly how he or she knows that all the possible rectangular pens have been drawn.</td>
<td>Draws all the possible rectangular pens.<br><br>Explains clearly how he or she knows that all the possible rectangular pens have been drawn.<br>Example:<br>The student uses patterns in a chart.
<table>
<tr><td>Length</td><td>Width</td></tr>
<tr><td>1 cm</td><td>18 cm</td></tr>
<tr><td>2 cm</td><td>9 cm</td></tr>
<tr><td>3 cm</td><td>6 cm</td></tr>
</table>
The student explains that a 3 cm by 6 cm rectangle is the same as a 6 cm by 3 cm rectangle. Therefore, all the possible rectangles with whole number dimensions are included in the chart.</td>
</tr>
<tr>
<td>Provide the student with triangular or isometric dot paper.<br>Present the student with the following problem:<br>"Marie measured a pattern block design with red trapezoids and said that the measure of the area was 6 trapezoids. Jaden measured the same design and said the area was 18 green triangles. Could both be correct? Why or why not? Draw a diagram and use words to explain your thinking. Use the triangular dot paper provided."</td>
<td>Guesses that both Marie and Jaden could be correct or incorrect but does not provide justification by drawing diagrams or explaining with words.</td>
<td>States that both Marie and Jaden are correct and proves it by drawing appropriate diagrams and explaining using correct mathematical language.<br><br>Example:<br>There are 3 green triangles that cover the same area as 1 red trapezoid. Therefore, 6 trapezoids have the same area as 18 green triangles because $3 \times 6 = 19$.</td>
</tr>
</table>

## B. Choosing Instructional Strategies

Consider the following general strategies for teaching volume (Van de Walle 2001):

- Access prior knowledge on using perimeter and area in the real world.
- Introduce volume by drawing on familiar and accessible contexts to illustrate uses of volume (NCTM 2000).
- Review the process used in developing understanding of perimeter and area and use a similar process in developing understanding of volume, stressing that the attribute changes but the process is similar:
  - Explain that the attribute to be measured is volume.
  - Check for conservation of volume; e.g., rearrange a given object and determine if the student realizes that the volume of the object remains unchanged.
  - Always estimate prior to comparing or measuring volumes.
  - Make direct comparisons; e.g., compare the volume of two boxes by placing one box inside the other box, if possible.
  - Estimate the volume of an object using nonstandard units of measure; e.g., sugar cubes. Use various techniques for estimating volume:
    - Referents—use a referent for the single unit of measure and iterate this unit mentally to obtain the estimate; e.g., use the size of the end of your smallest finger or a student's fore finger including the fingernail as a referent for 1 $cm^3$.
    - Chunking—estimate the volume of a smaller portion of an object initially and use this chunk to estimate the volume of the entire object; e.g., estimate the volume of a smaller section of a box and then multiply that answer by the number of these sections in the entire box.
    - Iteration—iterate a unit mentally or physically; e.g., use a single unit repeatedly to visually estimate the volume of an object.

    (Van de Walle and Lovin 2006)
  - Have the students share their strategies for estimating volume.
  - Accept a range of estimates—within 30% of the actual measure is reasonable (Van de Walle and Lovin 2006, p. 279).
  - Encourage the students to measure the volume after each estimate so that they develop a better sense of volume.
  - Use nonstandard units of measure that have the same attribute as the item being measured; e.g., use sugar cubes to measure volume of a box.
  - Make indirect comparisons using a nonstandard unit of measure that has the same attribute as the item being measured; e.g., use sugar cubes to measure the volumes of boxes for the purpose of comparison.
  - Measure the volume of an object using larger then smaller nonstandard units of measure to establish that the smaller the unit of measure, the more you need to measure the volume of a given object; e.g., more small wooden cubes are needed than larger wooden cubes to measure the volume of a given object.
  - Explain the need to use standard units to measure volume to facilitate communicating various areas globally.

- Measure the volume of a given object using an appropriate instrument with standard units of measure; e.g., use centimetre cubes that each have a volume of 1 $cm^3$ to measure the volume of a box.

- Integrate the strands by:
  - using patterns to develop understanding of volume
  - using multiplication and division of whole numbers in building or removing layers of cubes in a rectangular prism.

## C. Choosing Learning Activities

The following learning activities are examples of activities that could be used to develop student understanding of the concepts identified in Step 1.

**Sample Activities:**

**Teaching Conservation of Volume:**

1. **Shapes Made from Plasticine** (p. 15)
2. **Shapes Made from Cubes** (p. 15)

**Teaching Volume Measured in Nonstandard Units:**

1. **Concept of Volume** (p. 17)
2. **Using Nonstandard Units to Measure Volume** (p. 18)

**Teaching the Comparison of Volumes and the Justification for Cubic Units:**

1. **Focus on Comparing Volumes** (p. 20)
2. **Focusing on the Cubic Unit and Proportionality** (p. 21)

**Teaching the Use of Referents for $cm^3$ or $m^3$ in Estimating Volume and Then Finding the Actual Volume Using Standard Units:**

1. **Standard Units for Volume** (p. 23)
2. **Referents for Volume and Estimating Volume; Finding the Actual Volume** (p. 24)
3. **Estimate–Measure–Estimate–Measure Sequences** (p. 25)
4. **Estimate and Compare Volumes Using Standard Units** (p. 26)

**Teaching that Many Different Rectangular Prisms can be Constructed for a Given Volume ($cm^3$ or $m^3$):**

1. **Constructing Rectangular Prisms for a Given Volume Focusing on Layers** (p. 28)
2. **Constructing Rectangular Prisms for a Given Volume, Focusing on the Inverse Relationship Between the Area of the Base and the Height of the Prism** (p. 29)
3. **Frayer Model for Volume** (p. 31)

# Teaching Conservation of Volume

## Sample Activity 1: Shapes Made from Plasticine

Provide the students with two equal amounts of Plasticine; e.g., two congruent strips of Plasticine. Emphasize that the two strips of Plasticine contain the same amount of Plasticine. Have them mold the Plasticine into two different objects that have different shapes. Ask them if they think these two different objects occupy the same amount of space or have the same volume. Have the students immerse each of the objects in a graduated cylinder, record the water levels for each immersion and note their differences if any. Through discussion, have the students generalize that changing the shape of an object does not change the amount of space that the object occupies.

**Look For …**

Do students:

- ☐ state without hesitation that rearranged objects have the same volume as before the rearrangement?
- ☐ explain why the rearrangement of objects results in objects with the same volume?

## Sample Activity 2: Shapes Made from Cubes

Provide the students with sugar cubes. Have them divide the sugar cubes into two equal groups and make two different structures using half the sugar cubes in each structure. Ask the students if the two structures they created occupy the same amount of space; i.e., have the same volume. Guide discussion so that students count the number of sugar cubes in each structure and therefore understand that the two different structures occupy the same amount of space.

## Teaching Volume Measured in Nonstandard Units

## Sample Activity 1: Concept of Volume

Have the students share some everyday contexts in which it is necessary to fill objects and find the volume, such as filling a hole with dirt or a box with cubes. Draw on prior knowledge of perimeter and area to share ideas of where perimeter and area are used everyday. Have the students review what units are used to measure the perimeter and area of 2-D shapes. Ask if these units would be useful in measuring the amount needed to fill an object such as a box. Have the students suggest something that could be used to completely fill an object such as a box. Have marbles, sugar cubes, wooden cubes and other suitable objects available.

**Look For …**

Do students:

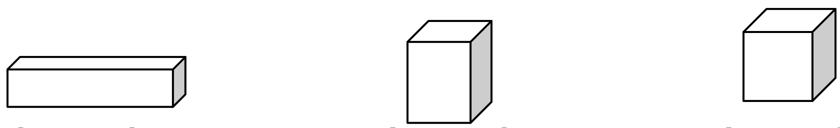
- ☐ distinguish among perimeter, area and volume?
- ☐ estimate volume using nonstandard units?
- ☐ repeatedly use the same sized unit to fill an object, such as a box, to find its volume?
- ☐ show with examples that more units are needed to measure a given volume when the units are smaller?

## Sample Activity 2: Using Nonstandard Units to Measure Volume

Provide each group of students with three boxes of different dimensions. Have marbles, packing peanuts, wooden cubes, multilink cubes, sugar cubes and other nonstandard units for volume available for the students to use. Present the following problem:

You are making candy to give your grandma for her birthday. Which of the boxes should you use for the candy so that your grandma will get the most candy?

Examples:



6 cm by 2 cm by 2 cm    2 cm by 2 cm by 4 cm    3 cm by 3 cm by 3 cm

Have the students explore different ways to find the volumes of the boxes, such as filling the boxes with objects such as marbles, packing peanuts, sugar cubes, wooden cubes or multilink cubes. Ask the students to share ideas for different ways to find the volumes of the boxes.

Have the students share their answers to the problem and discuss which method they think is the most accurate in finding the volumes. Guide the discussion to make the following generalizations about finding the volume of an object:

- repeatedly use the same size unit
- completely fill the box, leaving no gaps
- use a variety of ways to find volume, remembering that the smaller the unit used, the more of them will be needed to fill a given object.

# Teaching the Comparison of Volumes and the Justification for Cubic Units

## Sample Activity 1: Focus on Comparing Volumes

### Direct Comparison

This section adapted from W. George Cathcart, Yvonne M. Pothier and James H. Vance, *Learning Mathematics in Elementary and Middle Schools* (2nd ed.) (Scarborough, ON: Prentice Hall Allyn and Bacon Canada, 1997), p. 215. Used with permission of Pearson Canada.

Hold up two empty boxes (rectangular prisms). Choose the boxes so that one box fits inside the other box. Review comparison of length and area by focusing on attributes of the box that relate to length (e.g., the length of one edge) and area (e.g., the area of one face). Superimpose one box onto the other box to compare length and area. Then ask the students to predict which of the boxes has the greater volume; i.e., occupies more space. Have the students suggest how to compare the volumes of the two boxes. Through discussion, encourage the students to realize that one method could be to fit one box inside the other box to compare the volume of the two boxes.

Have the students suggest other objects that could be used to compare the volumes directly. Then guide the discussion to show that for many objects the volumes cannot be compared directly; therefore, indirect comparison must be used.

### Indirect Comparison

Hold up two empty boxes (rectangular prisms). Choose the boxes so that one box does not fit inside the other box. Ask the students for suggestions as to how they could compare the volumes of these two boxes. Guide the discussion to conclude that nonstandard units such as sugar cubes, Multilink cubes or packing peanuts could be used. By using multiples of the same nonstandard unit to fill one of the boxes, the volumes of the two boxes can be compared indirectly:

- Dump the nonstandard units from one box into the other to see if the volume of one box is less than, equal to or greater than the other box.
- Count the number of nonstandard units needed to fill each box and compare the two numbers.

**Look For …**

Do students:

- ☐ compare the volumes of two objects by using direct or indirect comparison?
- ☐ know the essential characteristics of a cube?
- ☐ explain clearly why the cubic unit is the preferred unit for measuring volume?
- ☐ show with examples that more cubic units are needed to measure a given volume when the cubic units are smaller?

## Sample Activity 2: Focusing on the Cubic Unit and Proportionality

Have the students make a nonstandard unit for volume that they could use to measure the volume of a paper clip box. Remind them that the box must be completely filled with the unit they choose and that they must repeat their unit at least four times to measure the volume of their paper clip box. They should be ready to justify their choice of unit. Provide Plasticine, play dough, paper and scissors.

Have the students compare their answers and share the units that they created to measure volume. Provide units shapes as spheres, rectangular prisms and cubes if students do not include these as their created units. Show a variety of different shaped rectangular prisms.

Guide the discussion to include the following ideas:

- Spheres are not useful units in measuring volume because they leave gaps.
- Rectangular prisms do not leave gaps but there are many different shapes for rectangular prisms so saying a unit is shaped like a rectangular prism is not descriptive enough.
- Cubes do not leave gaps and there is only one shape for a cube so everyone knows the shape of a cubic unit.

**Look For …**

Do students:

- ☐ recognize the difference between the standard units for perimeter, area and volume?
- ☐ use a personal referent for estimating volume that relates well to the standard units?
- ☐ explain clearly how to estimate the volume of a given object?
- ☐ estimate the volume prior to measuring the volume and then compare the actual volume to the estimated volume?

Provide the students with cubes to measure the volume of a given box. Give some groups large cubes and other groups small cubes. Compare the answers for the volumes found. Generalize: to compare volumes the same sized unit of measure must be used; i.e., either small cubes or large cubes. Review the fact that the smaller the unit used to measure volume, the more of these units are needed.

# Teaching the Use of Referents for $cm^3$ or $m^3$ in Estimating Volume and Then Finding the Actual Volume Using Standard Units

## Sample Activity 1: Standard Units for Volume

Review the linear units (centimetre and metre) used to find the perimeter of 2-D shapes. Explain that these standard units of measure were used to find perimeter so that the perimeters of shapes could be compared and communicated clearly. Similarly, review the standard units for area. Connect the need for standard units in finding perimeter and area to the need for standard units in finding volume.

Review the discussion of using a cubic unit for volume in the previous activity and ask the students to suggest standard units for measuring volume. Through discussion, conclude that the standard units used for volume are $cm^3$ and $m^3$. Write the symbols and explain that 1 $cm^3$ is read as "one cubic centimetre" **not** "one centimetre cubed." One cubic centimetre is a measure for the volume of a variety of shapes; whereas one centimetre cubed is a cube that is 1 cm on each edge.

Provide the students with centicubes and explain that the volume of each cube is 1 $cm^3$. Have the students use their centimetre rulers to measure the edge of each cube to verify that each cube is 1 $cm^3$.

## Sample Activity 2: Referents for Volume and Estimating Volume; Finding the Actual Volume

Review the referents used for centimetre (e.g., the width of the pinky finger) and metre (e.g., the distance from the teacher's finger tip to his or her opposite shoulder). Ask the students to suggest a suitable referent for 1 $cm^3$ and explain why they think it would work.

Have the students use their referents for 1 $cm^3$ to estimate the volume of a box in cubic centimetres. Then have them find the volume of the box by filling it with centicubes and compare their answers to their estimated answers.

Similarly, discuss possible referents for 1 $m^3$ after the students make a cube on the floor that is one metre on each edge using metre sticks taped together. Have the students use their referents and estimate the volume of air in the classroom. Use the cubic metre to find the volume of air in the room. The actual volume may not be that accurate because using the cubic metre repeatedly to find the volume is very cumbersome and difficult.

## Sample Activity 3: Estimate–Measure–Estimate–Measure Sequences



Select pairs of objects that are related or close in measure but not the same; e.g., two boxes of different sizes. Have the students estimate the volume of the first object and then measure its volume using standard units of measure. Then have the students estimate the volume of the second object and check the estimate by measuring its volume. With this sequence of estimate and then measure, students have practised applying their knowledge about estimating the volume of one object to estimating the volume of another object and thereby improve their estimating skills.

## Sample Activity 4: Estimate and Compare Volumes Using Standard Units

This activity adapted from John A. Van de Walle and LouAnn H. Lovin, *Teaching Student-Centered Mathematics: Grades 3–5*, 1e (p. 267). Published by Allyn and Bacon, Boston, MA. Copyright © 2006 by Pearson Education. Reprinted by permission of the publisher.

Provide the students with pairs of rectangular prisms that are folded up from poster board, such as the following:

First pair: 2 cm by 4 cm by 3 cm, 2 cm by 6 cm by 2 cm
Second pair: 2 cm by 2 cm by 5 cm, 4 cm by 3 cm by 3 cm

Have the students estimate which rectangular prism in each pair has the greater volume, or if they have the same volume, and explain their thinking.

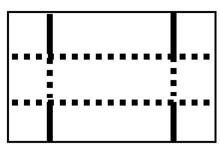
Provide the students with centicubes to fill their boxes to find their volumes and have them record the volumes using a number and the unit, $cm^3$. Have the students decide which rectangular prism in each pair has the greater volume, or if they have the same volume, and explain their thinking. Encourage the students to share their ideas.

**Look For …**

Do students:

- ☐ count the number of centicubes needed to fill each rectangular prism and then compare these numbers?
- ☐ use more than one way to compare the volume of different objects?

Make small boxes by starting with a rectangle and drawing a square on each corner. Cut on the solid lines and fold the box up, wrapping the corner squares to the outside and tape or glue them to the sides.



(Van de Walle and Lovin 2006, p. 267)

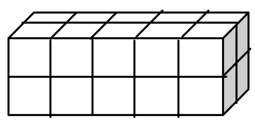
# Teaching that Many Different Rectangular Prisms can be Constructed for a Given Volume ($cm^3$ or $m^3$)

## Sample Activity 1: Constructing Rectangular Prisms for a Given Volume Focusing on Layers

This activity adapted from W. George Cathcart, Yvonne M. Pothier and James H. Vance, *Learning Mathematics in Elementary and Middle Schools* (2nd ed.) (Scarborough, ON: Prentice Hall Allyn and Bacon Canada, 1997), p. 215. Used with permission of Pearson Canada.

Provide the students with nonstandard units such as Multilink cubes or standard units such as centicubes.

Have the students use the cubes to build the rectangular prism that is pictured here: 5 units long, 2 units wide and 2 units high. Encourage the students to build one layer that is 5 units long and 2 units wide. Then superimpose another identical layer to make 2 layers showing that the height is 2 units. Building layers, in constructing rectangular prisms provides a strong basis for understanding the formula for finding the volume of rectangular prisms in Grade 6.



Instruct the students to find the volume of the prism. If necessary, encourage the students to find the volume by counting the number of cubes used to build the prism: first the number of cubes in one layer and then the total number of cubes in all the layers in the structure.

Have the students rearrange the cubes to make another prism with the same volume but different dimensions and encourage them to share their work with others.

Have the students build other rectangular prisms with different dimensions and record the volume of each one.

**Look For …**

Do students:

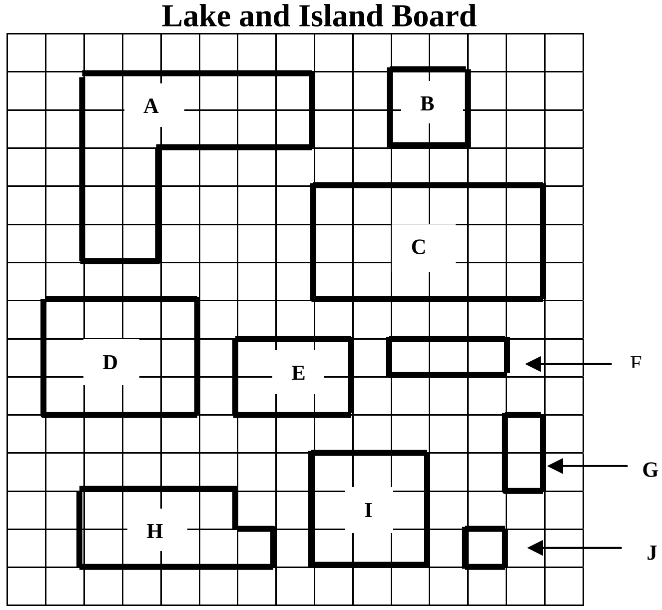
- ☐ build one layer and then decide how many other layers are needed to construct various rectangular prisms with the same volume?
- ☐ draw diagrams and record the data in the chart to explain the various rectangular prisms that can be constructed with a given volume?
- ☐ demonstrate conservation of volume as they create different rectangular prisms for a given volume?

## Sample Activity 2: Constructing Rectangular Prisms for a Given Volume, Focusing on the Inverse Relationship Between the Area of the Base and the Height of the Prism

**Lake and Island Board**
A lake and island board uses one square centimetre as the basic unit for measuring area. This board is used to explore concepts of perimeter, area and volume. The focus of this activity is on volume, with special emphasis on the inverse relationship between the area of the base and the height of a rectangular prism.

Provide the student with copies of the lake and island board shown below without the centimetre grid. Make a transparency of the board without the centimetre grid to use for discussion with the whole class.

**Lake and Island Board**




Provide each group of students with at least 12 centicubes. Suggest that each of these centicubes represents a suite in an apartment building. The length, width and height of the apartments refer to the dimensions.

You are to build a 12-suite apartment building on each of the following islands:

B, D, E and F. Record the area of each island, the dimensions of each apartment and the volume of each apartment in the chart below.

| **Island** | **Area** ($cm^2$) | **Length** (cm) | **Width** (cm) | **Height** (cm) | **Volume** ($cm^3$) |
|---|---|---|---|---|---|
| **B** | | | | | |
| **D** | | | | | |
| **E** | | | | | |
| **F** | | | | | |

a. How does the height of the 12-suite apartment building on island E compare with the height of the 12-suite apartment building on island F? Does this comparison have any relation to the areas of the two islands? If so, what is this relation?

b. Compare the heights of the 12-suite apartment buildings on islands D and E.

c. Build an apartment building of the same size on island F. Make note of the dimensions on the chart.

d. If you were to build a 12-suite apartment on each of islands B and G, which apartment would be higher? How many times higher would it be?
   Now do the construction and test your guess. Make note of the dimensions in the chart.

e. If you were to build a 12-suite apartment building on islands V and J, which apartment building would be higher? How many times higher would it be? Now do the construction and test your guess. Make note of the dimensions in the chart.

f. Explain how the volume of each apartment building is related to its dimensions.

g. Suppose you build a model of an apartment building that is 5 cm high to completely cover island C. What would the volume of this model be? Explain your thinking.

h. Suppose you build a model of an apartment building that completely covers island I and it has a volume of 72 cubic centimetres. How high would you have to make your model? Explain your thinking.

**Look For …**

Do students:

- ☐ construct different rectangular prisms, given the size of the base?
- ☐ explain how the height of each prism with the same volume differs when the size of the base changes?
- ☐ predict the height of rectangular prisms with the same volume when the base is given?
- ☐ explain how the volume of a rectangular prism is related to its dimensions?
- ☐ apply the knowledge derived by examining patterns in volume to solve problems using larger volumes?

## Sample Activity 3: Frayer Model for Volume

Provide the students with a template for the Frayer Model and have them fill in the sections individually or as a group to consolidate their understanding of volume.

A sample Frayer Model follows.

**Look For …**

Do students:

- ☐ define volume in their own words?
- ☐ describe the essential characteristics of volume?
- ☐ create a problem that applies the concept of volume?
- ☐ use visuals to show volume?
- ☐ provide examples of where volume is used in the real world?
- ☐ provide non-examples of volume in the real world; i.e., distinguish among perimeter, area and volume examples?

# Frayer Model for Volume

## Definition

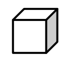
Volume is the amount of space occupied by an object.

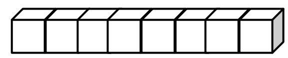
## Characteristics

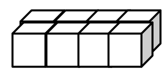
- The volume of an object remains the same when the object is rearranged.
- Volume can be measured in nonstandard or standard units for volume.
- The smaller the unit of measure, the greater the number of units needed to measure the volume of a given object.
- When comparing volumes, the same units must be used.
- Standard units for volume include $cm^3$ and $m^3$.
- For a given volume, there is usually more than one different rectangular prism than can be constructed.

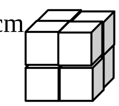
## Real-life Problem and Visual Representation

Kim wants to make a box in the shape of a rectangular prism with a volume of 8 $cm^3$ to hold her collection of beads. What are the different boxes that she could make if the lengths, widths and heights are whole numbers?

 = 1 $cm^3$

1 cm by 1 cm by 8 cm 

1 cm by 2 cm by 4 cm 

2 cm by 2 cm by 2 cm 

Kim could make 3 different boxes as shown above.

## Volume

## Examples

Volume is used in the following:

- sand in a sandbox
- dirt in a truckload
- boxes in a large container
- grain in an elevator
- air in a room.

## Non-examples

Volume is not used in the following:

- fencing around a garden
- lace around a tablecloth
- painting walls
- tiling floors
- covering countertops.

Format adapted from D. A. Frayer, W. C. Frederick and H. J. Klausmeier, *A Schema for Testing the Level of Concept Mastery*. (Working Paper/Technical Report No. 16) (Madison, WI: Research and Development Center for Cognitive Learning, University of Wisconsin, 1969). Adapted with permission from the Wisconsin Center for Education Research, University of Wisconsin-Madison.

More activities on volume are available on pages 160–161 of the *Diagnostic Mathematics Program, Division II, Measurement*.

## Step 4: Assess Student Learning

### Guiding Questions

- Look back at what you determined as acceptable evidence in Step 2.
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

### Sample Assessment Tasks

In addition to ongoing assessment throughout the lessons, consider the following sample activities to evaluate students' learning at key milestones. Suggestions are given for assessing all students as a class or in groups, individual students in need of further evaluation, and individual or groups of students in a variety of contexts.

### A. Whole Class/Group Assessment

**Estimating and Measuring Volume of Right Rectangular Prisms**
In this assessment task, the student will demonstrate understanding of estimating and measuring the volume of right rectangular prisms, focusing on the standard unit, cubic centimetres ($cm^3$). There are two parts to this assessment task.

**Part A:** Given a box, the student will estimate the volume of the box using a personal referent and describing the process. The students will *not* use centicubes to make the estimate; therefore, it is necessary that the students submit their work on Part A *before* they start work on Part B.

**Part B:** Given the dimensions, the students will construct a right rectangular prism (box) using centicubes or another method and explain the process. Then the students will record the volume of the right rectangular prism constructed and explain how the volume was found. Finally, the students will construct at least two more right rectangular prisms with the same volume and explain the process used.

Materials required: Part A: small labelled boxes, Part B: centicubes

Each student will:

- estimate the volume of a right rectangular prism and record the estimate using a number and a unit that matches the attribute of volume
- explain his or her choice of personal referent and the process used to estimate the volume of a rectangular prism
- construct a right rectangular prism, given the dimensions, and explain the process used
- measure and record the volume of a right rectangular prism and explain the process used to find the volume
- construct at least two more right rectangular prisms with the same volume and explain the process used.

It is important that students not have the centicubes when estimating the volume of the labelled boxes. They are to use a personal referent for a cubic centimetre and must visualize the size of a cubic centimetre when making the estimate. Once the estimate is made and handed in to the teacher, then the centicubes are distributed for Part B of the assessment task.

When estimating the volume of the box in Part A, the students should be within 30% of the actual volume to have their performance assessed as "adequate."

Early finishers can estimate the volume of other boxes provided in Part A. They can also find the volume of the rectangular prism in Part B if the dimensions of are doubled; i.e., the length is 8 cm, the width is 4 cm and the height is 6 cm. Then they can compare this volume with the volume they found for the original right rectangular prism that is 4 cm by 2 cm by 3 cm.

**Estimating and Measuring Volume of Right Rectangular Prisms** – Student Assessment Task

Part A – Estimating Volume

Choose a labelled box from the set of boxes.

a) Write the name of the label below.

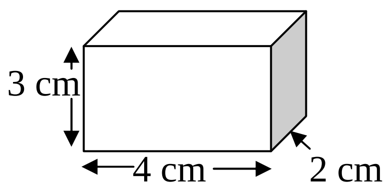
b) Estimate the volume of the box by using a personal referent. Explain your choice of referent and the process used to make the estimate.

**Give Part A to your teacher <u>before</u> you begin Part B.**

Part B – Measuring, Recording and Representing Volume

1. Use centicubes or another method to make a right rectangular prism (box) with the following dimensions:

   Length: 4 cm
   Width: 2 cm
   Height: 3 cm




   Explain with words and diagrams how you constructed the rectangular prism (box).

2. Record the volume of the right rectangular prism (box) described in Part B, Question #1. Explain how you found the volume using words and diagrams.

3. Construct at least two more right rectangular prisms (boxes) that have the same volume as the right rectangular prism (box) described in Part B, Question #1 but have different dimensions. Explain how you found the volume of each right rectangular prism (box) using words and diagrams.

Student ______________________________

# SCORING GUIDE
## Estimating and Measuring Volume of Right Rectangular Prisms

| Level / Criteria | 4 Excellent | 3 Proficient | 2 Adequate | 1 Limited * | Insufficient / Blank * |
|---|---|---|---|---|---|
| **Estimating volume** <br><br> **Part A** | Estimates within 10% of the actual volume of the box using a personal referent and correct units. Explains clearly and in detail the choice of referent and the process used to make the estimate. ☐ | Estimates within 20% of the actual volume of the box using a personal referent and correct units. Explains the choice of referent and the process used to make the estimate. ☐ | Estimates within 30% of the actual volume of the box using a personal referent and correct units. Provides limited explanation of the choice of referent and the process used to make the estimate. ☐ | Estimates outside of 30% of the actual volume of the box and may not include the correct units. Makes no effort to explain the choice of personal referent or the process used to make the estimate. ☐ | No score is awarded because there is insufficient evidence of student performance based on the requirements of the assessment task. ☐ |
| **Constructing a right rectangular prism** <br><br> **Part B** <br><br> **Question #1** | Constructs a right rectangular prism using the dimensions that are given and explains clearly and in detail with words and diagrams how the construction was done. ☐ | Constructs a right rectangular prism using the dimensions that are given and explains with words and diagrams how the construction was done. ☐ | Constructs a right rectangular prism using the dimensions that are given and explains with words or diagrams, but not necessarily both, how the construction was done. ☐ | Constructs a right rectangular prism using most of the dimensions that are given and explains vaguely with words or diagrams, but not both, how the construction was done. ☐ | No score is awarded because there is insufficient evidence of student performance based on the requirements of the assessment task. ☐ |
| **Measuring and recording the volume of a right rectangular prism** <br><br> **Part B** <br><br> **Question #2** | Records the volume of the right rectangular prism as 24 $cm^3$ and explains clearly and in detail with words and diagrams how the volume was found. ☐ | Records the volume of the right rectangular prism as 24 $cm^3$ and explains with words and diagrams how the volume was found. ☐ | Records the volume of the right rectangular prism as 24 $cm^3$ and provides limited explanation with words or diagrams, but not necessarily both, how the volume was found. ☐ | Records the volume of the right rectangular prism as a number close to or equal to 24 but not the unit, $cm^3$, and explains vaguely with words or diagrams, but not both, how the volume was found. ☐ | No score is awarded because there is insufficient evidence of student performance based on the requirements of the assessment task. ☐ |
| **Constructing more than one right rectangular prism for a given volume** <br><br> **Part B** <br><br> **Question #3** | Constructs three or more right rectangular prisms with a volume of 24 $cm^3$ and explains clearly and in detail with words and diagrams how the constructions were done. ☐ | Constructs two right rectangular prisms with a volume of 24 $cm^3$ and explains with words and diagrams how the constructions were done. ☐ | Constructs one right rectangular prism with a volume of 24 $cm^3$ and provides a limited explanation with words or diagrams, but not necessarily both, how the construction was done. ☐ | Constructs one right rectangular prism with a volume close to or equal to 24 $cm^3$ and explains vaguely with words or diagrams, but not both, how the construction was done. ☐ | No score is awarded because there is insufficient evidence of student performance based on the requirements of the assessment task. ☐ |

***** When work is judged to be limited or insufficient, the teacher makes decisions about appropriate intervention to help the student improve.

## B. One-on-one Assessment

**Estimating Volume**

If the student has difficulty estimating volume using standard units of measure, provide the student with centicubes and explain that each centicube is 1 cubic centimetre or 1 $cm^3$ because it is 1 cm long, 1 cm wide and 1 cm high. Assist the student in deciding on a personal referent for 1 $cm^3$. For example, the end of a student's forefinger including most or all of the fingernail is usually about 1 $cm^3$. Have the students suggest objects that would have a volume of about 1 $cm^3$; e.g., a small gumball or the eye of a teddy bear.

**Measuring and Recording Volume**

Provide the student with objects of different sizes, including rectangular prisms (boxes), and have the student estimate the volume in cubic centimetres. Encourage the student to use his or her personal referent to estimate and also explain the process used in estimating. Immediately after estimating, have the students measure the volume of the object by using centicubes to fill the boxes or by immersing the object in a graduated cylinder containing water. Explain that 1 mL of water is the same amount as 1 $cm^3$. Remind the student that volume must be recorded using a number and a correct unit to match the attribute of volume.

If the student has difficulty measuring and recording the volume of objects, use rectangular prisms (boxes) with various dimensions and have the student fill the boxes with centicubes. Encourage the student to completely fill the bottom layer and then decide how many layers are needed to completely fill the box. Then the student can fill the box with centicubes layer by layer or they can multiply the number of centicubes in one layer by the number of layers to get the volume of the box. See page 160 of the *Diagnostic Mathematics Program, Division II, Measurement* for a template for making boxes of various sizes.

**Constructing Right Rectangular Prisms**

If the student has difficulty constructing many right rectangular prisms for a given volume, provide him or her with Multilink cubes, sugar cubes or centicubes and start with a small number of cubes, such as eight. Suggest that the student make a right rectangular prism (box) using the eight cubes. Have the student describe the structure made by listing the three dimensions. Then ask the student to make a different right rectangular prism with the eight blocks and list the new dimensions. If necessary, start the structure for the student and have him or her finish it once the first layer is created. Have the student draw diagrams to represent each of the structures made and include the dimensions in a chart. Encourage the student to look for and describe patterns that can be seen from the number listed in the chart. Continue with larger numbers.

Further one-on-one assessment for finding volume without formulas is provided in the structured interview and the written assessment task on pages 71–77 of the *Diagnostic Mathematics Program, Division II, Measurement*.

## C. Applied Learning

Provide opportunities for the students to use their understanding of volume in a practical situation and notice whether or not this understanding transfers. For example, have the students find the volume of an object such as a small, empty box in the shape of a rectangular prism. Does the student:

- estimate the volume of the object prior to finding the actual volume?
- use units that match the attribute of volume?
- use nonstandard (e.g., sugar cubes) or standard units (e.g., centimetre cubes) to measure the volume of the object by filling it?
- use units of an appropriate size to measure the volume?
- write the volume using a number and the unit chosen?
- compare the estimated volume with the volume found by using nonstandard or standard units?
- apply this concept to other situations, such as finding the volume of different objects that can be filled?

## Step 5: Follow-up on Assessment

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction?

### A. Addressing Gaps in Learning

To improve their estimating skills, have the students always estimate before measuring and then compare their measurements to the original estimates. As they become more familiar with the units used in volume measurements, the students will have a better sense of estimating the volume in the required units. Remind the student to use referents and/or chunking when estimating volume. Have the students share their estimates and strategies for estimating. Begin by having the students estimate by comparing the volumes of two different objects. Then have them estimate the volume of objects. Accept a range of estimates and narrow the range as the students' estimating skills improve.

Conservation of volume develops as students mature. Continue to provide opportunities for students to compare the volumes of objects that have different shapes with the same size (volume) and have them communicate their thinking. Provide objects that are easily rearranged so that one object can be rearranged to look exactly like the other object. Use Multilink cubes as they are easily rearranged to make a variety of objects with the same volume. Encourage the students to rearrange them as needed.

Students who have difficulty repeating the same unit when measuring volume should have ample opportunity to explore using manipulatives such as sugar cubes, wooden cubes and centimetre cubes. Encourage the students to manipulate the concrete materials and explain how the unit is repeated when finding the volume.

If students have difficulty constructing different rectangular prisms for a given volume, use smaller structures made with Multilink cubes initially. Encourage the student to use the Multilink cubes to make one layer of cubes for the base of the rectangular prism and then build congruent layers for a given volume. If the student works randomly and misses some rectangular prisms with a given volume, suggest that he or she use a pattern to help solve the problem and record it in a chart. For example, the possible rectangular prisms with a volume of 12 $cm^3$ could be represented in a chart such as:

| Length | Width | Height |
|---|---|---|
| 1 | 1 | 12 |
| 1 | 2 | 6 |
| 1 | 3 | 4 |
| 2 | 2 | 3 |

Remind the student that changing the orientation of the rectangular prism does not result in a new rectangular prism; i.e., a length of 3 cm, width of 2 cm and height of 2 cm is the same rectangular prism as one with a length of 2 cm, a width of 2 cm and a height of 3 cm.

## B. Reinforcing and Extending Learning

Students who have achieved or exceeded the outcomes will benefit from ongoing opportunities to apply and extend their learning. These activities should support students in developing a deeper understanding of the concept and should not progress to the outcomes in subsequent grades.

Consider strategies such as:

- Provide tips for parents on helping their children to estimate and find volumes of objects. For example:
  - Ask the child to estimate the number of candies of the same size and shape needed to fill a candy jar. Have the child explain how he or she estimated the volume using candies as the unit for volume.
  - Ask the child to estimate the volume of dirt needed to fill a large hole in the back yard or to make a truckload. Have the child explain how he or she knows.
  - Ask the child to make or describe the dimensions of a box that will hold a given number of smaller boxes all the same size and shape.
- Provide the students with the following pair of rectangular prisms: 3 cm by 4 cm by 2 cm and 2 cm by 2 cm by 6 cm. Ask the students to compare the two prisms and decide if they have the same volume or if the volume of one prism is greater than the volume of the other prism. Encourage the student to use centimetre cubes to prove his or her answer.
- Pose the following problem to the students:
  Katrina said that if you double the dimensions of a rectangular prism then you double the volume. Do you agree with Katrina? Why or why not?
- Box Measuring

  Provide the students with centimetre grid paper, centimetre cubes and scissors. Present the following problem:
  If you had a piece of 20 by 20 centimetre grid paper, you could cut a square the same size from each corner and fold up what's left to make a box.
  - How many different boxes could you make using this method, making each from a different piece of 20 by 20 centimetre grid paper? Make the boxes to show your answer.
  - Which of these boxes holds the most? Explain how you found the volume of each box to compare the volumes.

- Make a Candy Box
  Present the following problem:
  Chrystal and Kodie made a lot of candy for their friends. Each piece is in the shape of a cube, 2 cm on each edge. They want to put the candies into boxes. Each box will hold 8 pieces of candy.
    i. Make a box using centimetre grid paper that will just hold the 8 pieces of candy. Explain your thinking.
    ii. Make at least one other box with different dimensions that will just hold the 8 pieces of candy. Explain your thinking.
    iii. Design 3 different boxes that will each hold 12 pieces of candy. For each box, the candy should be stacked in a different arrangement. The sides for all the boxes must be rectangles (squares can be used because a square is a rectangle). Remember, each piece of candy is a cube that is 2 cm on each edge.

  Adapted from National Council of Supervisors of Mathematics, *Great Tasks and More!* (Reston, VA: NCSM, 1996), pp. 26–47, 26–53.

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